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An Example in Complex L^1 Approximation

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Let (Ω, μ) be a nonatomic measure space, K either the set R of real numbers or the set C of complex numbers, and $L^1(\mu; K)$ the space of K-valued integrable functions on Ω. Suppose M is a finite-dimensional subspace of $L^1(\mu; K)$; let S be the set of elements in $L^1(\mu; K)$ that have more than one best approximation from M, and let S' be the set of elements whose set of best approximations from M contains a relatively open subset of M. In the real case, Havinson and Romanova have shown that both S and S' are dense in $L^1(\mu; R)$. In the present paper, a result of Kripke and Rivlin is used to show that, for $K = C$, μ finite, and M equal to the subspace of constant functions, S is not dense and S' is empty, thus, answering a question posed by Havinson and Romanova.

Let (Ω, μ) be a nonatomic measure space and E an arbitrary Banach space with norm $|\cdot|$. Let $\mathscr{L}^1(\mu; E)$ denote the space of Bochner integrable functions with seminorm $\int_\Omega |f(s)|\, d\mu(s)$; let $L^1(\mu; E)$ be the associated Hausdorff space. Suppose M is any finite-dimensional subspace of $L^1(\mu; E)$. In [5, 6], it was shown that, even though M fails to be Chebyshev, it is (in the terminology of Garkavi [1]) almost Chebyshev (i.e., the set S of elements that do not have a unique best approximation is of the first category, thus, the set of elements in $L^1(\mu; E)$ with a unique best approximation is dense and of the second category). When $E = R =$ the set of real numbers, this result was obtained (independently) by Havinson and Romanova [3]. They also showed that, for $E = R$, the set S' of elements that have "full nonuniqueness" (i.e., the set S' of elements for which the set of best approximations contains a relatively open subset of the subspace M) is also dense. They remarked that, for $E = C =$ the set of complex numbers, it could be shown that M is almost Chebyshev; they asked whether S' is also dense in this case. We show here that, with μ finite and M equal to the subspace of constant functions, the set S fails to be dense and the set S' is empty.

The crux of the matter is contained in a result due to Kripke and Rivlin [4, Theorem 2.3] (see also [2]). This theorem is stated by them for a σ-finite measure space (Ω, μ), but this is not actually needed in their proof. (In their notation, the disjoint sets $F(\lambda)$, $\lambda \in R$, are each contained in $R(f)$, which, as the support of an integrable function, must be σ-finite. Hence, at most a countable number of the sets $F(\lambda)$, $\lambda \in [-1, 1]$, can have positive measure. This is the only place in their proof that σ-finiteness is used.) We state the theorem in the following manner:

LEMMA (Kripke–Rivlin). *Let M be a subspace of $L^1(\mu; C)$. Suppose f has two distinct best approximations, say p_1 and p_2, from M. Let $f^* = f - \frac{1}{2}(p_1 + p_2)$ and $q = \frac{1}{2}(p_1 - p_2)$. Then, there is a nonzero $\lambda \in [-1, 1]$ and a real-valued function g, with $g(x) \in (-1, 1)$, satisfying, for almost all $x \in \Omega$,*

$$\lambda q(x) = g(x) f^*(x).$$

EXAMPLE. Let μ be finite and let M be the subspace of $L^1(\mu; C)$ consisting of the constant functions. Then, the set of elements with more than one best approximation is not dense: For, apply the above lemma to M. Then, for any element f with more than one best approximation, there exist complex numbers $p_1 \neq p_2$, a real number $\lambda \neq 0$, and a real-valued function g satisfying

$$\frac{\lambda}{2}(p_1 - p_2) = g(x)\left[f(x) - \frac{1}{2}(p_1 + p_2)\right]$$

for almost all $x \in \Omega$. Hence, $g(x) \neq 0$ for such x. Therefore,

$$f(x) = \frac{\lambda}{2g(x)}(p_1 - p_2) + \frac{1}{2}(p_1 + p_2)$$

for almost all $x \in \Omega$. Since λ and $g(x)$ are real numbers, we conclude that, for almost all $x \in \Omega$, $f(x)$ lies on the line in the complex-plane generated by p_1 and p_2. Clearly, the set of functions f with this property fails to be dense. It is also clear from this argument that the set of best approximations to f is contained in the line generated by p_1 and p_2. Thus, even though M is not Chebyshev, there is no element in $L^1(\mu; C)$ whose set of best approximations contains a relatively open subset of M. That is, S' is empty.

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